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Electromagnetic Diffraction by Imperfectly Dielectric Wedges

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1. INTRODUCTION

The first solution to a diffraction problem which has been formulated as a boundary value problem for the Maxwell equations, is that of the Sommerfeld half-plane problem. It consists essentially in solving the Helmholtz equation in the plane complement of a semi-infinite straight line on which the boundary values of the solution are prescribed. H. Poincaré [31] and A. Sommerfeld [35] were the first to work on this problem. Since then it has been the subject of considerations by different authors, representative of whom the following may be cited: W. Magnus [24], C. J. Bouwkamp [3], E. T. Copson [5], T. B. A. Senior [34], H. Hochstadt [11], M. Papadopoulos [30], and A. E. Heins [8]. The generalization of the problem which consists in replacing the half-plane by a perfectly conducting wedge, is treated in papers by T. J. Bromwich [4], H. M. MacDonald [23], J. B. Keller and A. Blank [15], G. Herglotz [9], F. Oberhettinger [29], S. Blume [2], and M. Roseau [33]. Some of these authors have also considered the Neumann problem of the Helmholtz equation. Supposing a large but finite conductivity of a dielectric wedge the diffraction problem may be formulated as a Robin problem. Papers dealing with this subject were published by D. S. Jones and F. B. Pidduck [13], L. B. Felsen [6], F. C. Karal and S. N. Karp [14], W. E. Williams [39], H. A. Lauwerier [22], and G. D. Maliuzhinetz [26]. If any finite conductivity is permitted, the continuity conditions of the Maxwell theory lead to the consideration of a transmission problem. This is done in the case of a right-angled wedge in papers by J. Radlow [32], N. H. Kuo and M. A. Plonus [17], and E. A. Kraut and G. W. Lehman [16]. The reasoning used in [32] is not correct, as it was pointed out in [16]. The existence proof for the solution of the problem in [16] is based on a restrictive condition, analogous to one used in [27]. No restriction of that kind was imposed in the papers [19] and [20] of the present author.

This paper, which contains essential parts of [19], deals with a transmission problem for the two-dimensional Helmholtz equation. It corresponds to the diffraction of electromagnetic waves by a system of imperfectly dielectric wedges with common edge, assuming arbitrary positive constants of electric conductivity and of dielectric permittivity in every wedge. The problem is treated by using the integral equation method, as it was done in papers by W. Sternberg [36], H. Freudenthal [7], W. D. Kupradse [18], C. Müller [28], R. B. Barrar and C. L. Dolph [1], and P. Werner [38]. In addition to the requirements of the problems considered by the above authors, the wedge shape demands attention to its edge and its infinite boundary.

2. Formulation of the Problem

Let a system of half-planes with a common edge generate a decomposition of space into wedge-shaped domains which are assumed to be imperfect dielectrics of equal magnetic permeability. An incident primary field of transverse magnetic type is produced by a line source which is situated parallel to the edge of the wedges. The time dependance is supposed to be harmonic. In order to determine the electric intensity of the secondary field, the following problem must be solved:

Let the x-plane Q be divided into open sectors S_i with common apex where i belongs to the set $T(m)$ of natural numbers between 1 *and* $m \geqslant 2$. *Then a complex-valued function Φ, defined on the set-theoretic union of the sectors, is to be determined by the following requirements:*

1. *The restriction of Φ on S_i is twice continuously differentiable and is a solution of the Helmholtz equation:*

$$\Phi \in C^2(S_i) \qquad \textit{and} \qquad (\Delta + \mu_i)\, \Phi(x) = 0,$$
$$\mu_i := k_i^2 = \omega^2\epsilon_i + j\omega\eta_i \in \mathbb{C}^+, \qquad j^2 = -1,$$

$\mathbb{C}^+$ *denoting the open first quadrant in the complex plane* $\mathbb{C}$.

2. *The continuation of Φ to the closure of S_i is once continuously differentiable with the possible exception of the apex. If $x \in \partial S_1$ holds, Φ and its normal derivative are subject to the following transmission conditions:*

$$\Phi_+(x) - \Phi_-(x) = E^0(x)$$

and

$$\partial\Phi_+(x)\,(n) - \partial\Phi_-(x)\,(n) = \partial E^0(x)\,(n).$$

The right-hand sides of the above equations are given by the electric intensity of the primary field supposing that the line source is located in the interior of the *wedge being generated by the sector* S_1. *The plus sign denotes boundary values from the exterior of* S_1, *while the minus sign corresponds to convergence from the interior. If* $x \in \partial S_i$ *and* $1 \neq i \in T(m)$ *hold, the same conditions are valid, the right-hand sides being zero.*

3. *There exists* $\alpha \in [0, 1)$ *such that, uniformly for all directions* x_0, *edge conditions hold:*

$$\Phi(\rho x_0) = O(1) \qquad \textit{and} \qquad \operatorname{grad} \Phi(\rho x_0) = O(\rho^{-\alpha}),$$

ρ *approaching zero.*

4. *There exists* $\beta > 0$ *such that, uniformly for all directions, radiation conditions hold:*

$$\Phi(\rho x_0) = O(e^{-\beta\rho}) \qquad \textit{and} \qquad \operatorname{grad} \Phi(\rho x_0) = O(e^{-\beta\rho}),$$

ρ *approaching infinity.*

The primary field is supposed to be known.

Concerning this problem the existence and the uniqueness of a solution will be proved. Using Laplace transform techniques we shall start deriving an integral equation which is shown to be uniquely solvable under no additional restriction by the Banach fixed-point theorem; then we shall verify, that the solution obtained satisfies all requirements of the problem quoted, thus proving its unique solvability.

3. Derivation of the Integral Equation

From the conditions of the problem we deduce a functional equation for the Laplace transform of the solution sought. Inversion then leads to the integral equation. Proceeding like this we apply:

Lemma 1. *For all pairs* (u_1, u_2) *of complex numbers* u_1 *and* u_2, *restricted by the condition* $(\operatorname{Re} u_1)^2 + (\operatorname{Re} u_2)^2 < \beta^2$, *the following formula of Green's type is valid:*

$$\int_{S_i} [e^{-ux} \Delta\Phi(x) - \Phi(x)\, \Delta e^{-ux}]\, dx = \int_{\partial S_i} [e^{-us} \partial\Phi(s)\,(n) - \Phi(s)\, \partial e^{-us}(n)]\, ds. \tag{3.1}$$

The product ux *is defined as* $ux := u_1x_1 + u_2x_2$ *for* $x = (x_1, x_2)$ *and* $u = (u_1, u_2)$. *The outward directed normal with respect to* S_i, $i \in T(m)$, *is denoted by* n.

Using the Helmholtz equation the left-hand side of the equation (3.1) can be transformed into the product of a quadratic form and the Laplace transform of Φ restricted on S_i. The right-hand side of the formula contains the boundary value transforms of Φ and its normal derivative. As the equation holds for every $i \in T(m)$, the transmission conditions can be involved by taking the sum:

$$\sum n_i(u)\, \varphi_i(u) = z_1(u), \qquad i \in T(m). \tag{3.2}$$

The symbols in the last equation are defined as follows:

$$\begin{aligned} n_i(u) &:= u_1{}^2 + u_2{}^2 + \mu_i \\ \varphi_i(u) &:= \int_{S_i} e^{-ux}\Phi(x)\, dx \\ z_1(u) &:= \int_{\partial S_1} [e^{-us}E^0(s)\,(n) + E^0(s)\,\partial e^{-us}(n)]\, ds. \end{aligned} \tag{3.3}$$

The function z_1 is known. We transform the equation (3.2) taking into account:

LEMMA 2. *Let $\mu = k^2 \in \mathbb{C}$ be such, that $q := \operatorname{Im} k > 0$ is valid. Then the quadratic form $n(u) = u_1{}^2 + u_2{}^2 + \mu$ is different from zero for all values $u = (u_1, u_2) \in \mathbb{C} \times \mathbb{C}$ under the restriction of $(\operatorname{Re} u_1)^2 + (\operatorname{Re} u_2)^2 < q^2$.*

Lemma 2 allows us to divide both sides of the equation (3.2) by $n(u)$ for all pairs $u = (u_1, u_2) \in \mathbb{C} \times \mathbb{C}$, which are restricted by

$$(\operatorname{Re} u_1)^2 + (\operatorname{Re} u_2)^2 < \min(\beta, q).$$

Then the following functional equation results:

$$\sum \varphi_i = z_1 n_1^{-1} + (\mu_1 - \mu)\, z_1 n_1^{-1} n^{-1} + \sum (\mu - \mu_i)\, \varphi_i n^{-1}, \qquad i \in T(m). \tag{3.4}$$

The left-hand side of (3.4) is the Laplace transform of Φ on Q:

$$\begin{aligned} (L\Phi)\,(u) &:= \int_Q e^{-ux}\Phi(x)\, dx \\ &= \sum \varphi_i(u), \qquad i \in T(m). \end{aligned}$$

The first term on the right-hand side of (3.4) may be interpreted as the Laplace transform of E^0 on the complement of S_1 by the following reasoning: The formula (3.1), applied to E^0 on $Q - S_1$, leads to $- z_1(u)$; as E^0 is a solution of the Helmholtz equation on $Q - S_1$, the left-hand side is equal to the product of $-n_1(u)$ and the Laplace transform of E^0; if therefore $\Omega^0 : Q \to \mathbb{C}$ is defined by

$$\Omega^0(x) = E^0(x), \qquad \text{if} \qquad x \in Q - S_1$$

and

$$\Omega^0(x) = 0, \qquad \text{if} \qquad x \in S_1,$$

we deduce the following relation:

$$(L\Omega^0)(u) = z_1(u)\, n_1^{-1}(u).$$

The remaining additive terms on the right-hand side of the equation (3.4) consist of products, everyone of which being the Laplace transform of a convolution. In order to realize this we use:

LEMMA 3. *Let $F, G : Q \to \mathbb{C}$ be square-integrable, and their Laplace transform be absolutely convergent for $u \in \mathbb{C} \times \mathbb{C}$. Then the convolution $H := F * G : Q \to \mathbb{C}$ exists, is bounded, and uniformly continuous. Its Laplace transform converges absolutely for u, and the convolution theorem holds:*

$$(LH)(u) = (LF)(u) \cdot (LG)(u).$$

LEMMA 4. *Let $H_0^{(1)} : Q \to \mathbb{C}$ denote the Hankel function of the first kind of order zero: $x \mapsto H_0^{(1)}(k \mid x \mid)$. Then $H_0^{(1)}$ is integrable and square-integrable on Q. Its Laplace transform exists, and is holomorphic for all values $u \in \mathbb{C} \times \mathbb{C}$ restricted by $(\operatorname{Re} u_1)^2 + (\operatorname{Re} u_2)^2 < q^2$. The following equation holds:*

$$(LH_0^{(1)})(u) = 4jn(u), \qquad j^2 = -1. \tag{3.6}$$

Using the last two lemmas the second term on the right-hand side of the equation (3.4) can be interpreted in the following way. Let $\Omega^1 : Q \to \mathbb{C}$ be defined by

$$\Omega^1(x) := \frac{1}{4j}(\mu_1 - \mu)\,(H_0^{(1)} * \Omega^0)(x), \tag{3.7}$$

then the following equation holds:

$$(L\Omega^1)(u) = (\mu_1 - \mu)\, z_1(u)\, n_1^{-1}(u)\, n^{-1}(u).$$

In the same manner each of the remaining products on the right-hand side of the equation (3.4) is recognized as the Laplace transform of a convolution. Therefore by means of inversion we arrive at the following:

THEOREM 1. *The conditions of the problem formulated lead to the integral equation:*

$$\Phi = \Omega + U\Phi, \tag{3.8}$$

the function Ω *and the operator* U *being defined as follows:*

$$\Omega(x) := \Omega^0(x) + \Omega^1(x),$$

$$(U\Phi)(x) := \sum \frac{1}{4j} (\mu - \mu_i) (H_0^{(1)} * \Phi_i)(x) \tag{3.9}$$

$$= \sum \frac{1}{4j} (\mu - \mu_i) \int_{S_i} H_0^{(1)}(k \mid x - y \mid) \Phi(y) \, dy, \qquad i \in T(m),$$

and assuming that $\operatorname{Im} k > 0$ *holds for* $\mu = k^2 \in \mathbb{C}$.

In the following section we discuss, independently from its particular derivation, the family of integral equations (3.8) in spaces of square-integrable functions.

4. Solution of the Integral Equation

We start by investigating the equation (3.8) in the Hilbert space $L_2(Q)$ of complex-valued functions being square integrable on Q. The mapping Ω^0 belongs to L_2 because of the primary field properties. The same holds for the remaining additive terms on the right-hand side of the integral equation as implied by:

LEMMA 5. *For every* $G \in L_2(Q)$ *the convolution* $H := (1/4j) H_0^{(1)} * G$ *defines a square-integrable function with the following norm-estimate:*

$$\| H \| \leqslant C_0 \| G \| . \tag{4.1}$$

The constant $C_0 > 0$ *is the maximum of* $| - | \tau |^2 + \mu |^{-1}$ *with respect to* $\tau = (\tau_1, \tau_2) \in \mathbb{R} \times \mathbb{R}$.

In consequence of this lemma the operator U is defined on the space L_2 with the range of values in L_2. We now restrict the complex numbers μ in such a way that U becomes a contraction operator. For that purpose let $\Gamma: Q \to \mathbb{C}$ be defined by

$$\Gamma(x) = \sum (\mu - \mu_i) E_i(x), \qquad i \in T(m), \tag{4.2}$$

where E_i denotes the characteristic function of the sector S_i. Then the operator U may be written as follows:

$$U\Phi = \frac{1}{4j} H_0^{(1)} * \Gamma\Phi. \tag{4.3}$$

Using the equation (4.1) we conclude:

$$\| U\Phi \| \leqslant D_0 \| \Phi \| . \tag{4.4}$$

In this inequality the constant D_0 is given by:

$$D_0 := \max | \mu - \mu_i | \cdot C_0 , \qquad i \in T(m).$$

U is a contraction mapping if D_0 is less than 1. In order to arrive at this, we specialize $\mu = k^2 \in \mathbb{C}$, restricted by $\operatorname{Im} k > 0$, to: $0 < | \operatorname{Re} k | < \operatorname{Im} k$. Then C_0 is equal to $| \mu |^{-1}$ and D_0 is less than 1 if the following set is not empty:

$$K_0 := \{\mu \in \mathbb{C}\colon \operatorname{Re} \mu \leqslant 0, | \mu - \mu_i | < | \mu | , i \in T(m)\}. \tag{4.5}$$

The set K_0 is to be considered as part of the intersection of m half-planes in $\mathbb{C}$, each of which is fixed by the mid-perpendicular M_i of the line-segment between the origin and $\mu_i \in \mathbb{C}^+$, $i \in T(m)$. K_0 is unbounded and contains a sector which may be described in the following manner: Its apex is that intersection point between the imaginary axis and one of the mid-perpendiculars, the absolute value of which is a maximum; one side of the sector is part of the positive imaginary axis; the other side is parallel to that mid-perpendicular, whose angle of elevation with respect to the positive real axis is a minimum. By these arguments we conclude that for every $\mu \in K_0$, the corresponding integral equation (3.8) has a unique solution in L_2 in consequence of the Banach fixed-point theorem.

We show that the solution does not depend on μ, using a reasoning which involves:

LEMMA 6. *Let $f, g \in L_2$ be the Fourier–Plancherel transforms of F, $G \in L_2$. If $f \cdot g \in L_2$ is valid, then $F * G \in L_2$ and the convolution theorem holds.*

We now fix $\mu_0 \in K_0$ and the corresponding solution $\Phi_0 \in L_2$ of (3.8). The application of the Fourier–Plancherel transform leads to the equation (3.4) holding on the pair of imaginary axes. We change this equation into (3.2) and divide both sides by $n(u) = u_1^2 + u_2^2 + \mu$, where u is specialized on the imaginary axes and μ is chosen arbitrarily as in Lemma 2. Then the reasoning of Section 3 leads to the conclusion that Φ_0 solves the integral equation (3.8) for every $\mu = k^2 \in \mathbb{C}$, restricted by $\operatorname{Im} k > 0$. The fact that Φ_0 is also the only solution in the space L_2 of the family of integral equations (3.8), follows in the same manner. Summarizing the preceding result we arrive at:

THEOREM 2. *For every set of numbers* $\mu_i \in \mathbb{C}^+$, $i \in T(m)$, *there exists an unbounded subset* K_0 *of* $\mathbb{C}$ *which is such that for* $\mu \in K_0$ *the integral equation* (3.8) *has a unique square-integrable solution as a consequence of the Banach fixed-point theorem. The solution does not depend on* $\mu \in K_0$, *and satisfies all integral equations* (3.8); *moreover it is the only solution.*

The Neumann series corresponding to the resolvent operator of (3.8), can be interpreted as that power series, which represents the solution with dependence on the wave numbers μ_i, $i \in T(m)$. The number $\mu \in K_0$ is the center of the expansion in each variable.

We now discuss the solvability of the equation (3.8) in special spaces. Let $M_2(Q)$ be the subset of L_2 the elements of which are bounded on Q. Then $\Omega^0 \in M_2$ holds due to the properties of the primary field. The remaining convolution terms on the right-hand side of the equation (3.8) belong also to M_2 as implied by Lemma 3. As a consequence of this Φ is an element of M_2. It may also be concluded that the function $P_n: Q \to \mathbb{C}$, $n \in \mathbb{N}$, defined for every $x \in Q$ as the partial sum of the Neumann series, is bounded. Therefore the following sup-norm-estimate can be established:

$$\sup | P_n - \Phi | \leqslant \max | \mu - \mu_i | \cdot \tfrac{1}{4} \| H_0^{(1)} \| \cdot \| S_{n-1} - \Phi \| , \qquad i \in T(m). \tag{4.6}$$

It shows that the Neumann series of (3.8) converges uniformly on Q. The summary of the above reasoning is the following:

THEOREM 3. *The square-integrable solution* $\Phi: Q \to \mathbb{C}$ *of the equation* (3.8) *is bounded. The corresponding Neumann series is uniformly convergent on* Q *for every* $\mu \in K_0$.

By means of Theorem 3 it is possible to verify those requirements of the diffraction problem which include continuity and differentiability.

With respect to the radiation conditions formulated we discuss the solvability of the equation (3.8) in the following set of spaces: Let $L_2(Q; \sigma)$ be the subset of L_2 consisting of functions which are square-integrable on Q with the weight function $w: Q \to \mathbb{R}$ given as $w(x) = \exp(\sigma x)$ for $x \in Q$. We restrict the pair $\sigma = (\sigma_1, \sigma_2) \in \mathbb{R} \times \mathbb{R}$ by the condition: $0 \leqslant | \sigma | < \operatorname{Im} k_1 =: q_1$. The product σx is defined as $\sigma x := \sigma_1 x_1 + \sigma_2 x_2$ for $x \in Q$. We introduce a distance function d on $L_2(Q; \sigma)$ by defining:

$$d(F, G) := \| F - G \| + \| w(F - G) \| .$$

Then $L_2(Q; \sigma)$ becomes a complete metric space. Investigating the equation (3.8) in $L_2(Q; \sigma)$, we consider $k \in \mathbb{C}$ under the restriction $\operatorname{Im} k = q \geqslant q_1$. Further we observe that Ω^0 is subject to the following radiation condition as ρ approaches infinity:

$$\Omega^0(\rho x_0) = O(e^{-q_1 \rho}). \tag{4.7}$$

Therefore $\Omega^0 \in L_2(Q; \sigma)$ holds. As a consequence of this Ω^1 is also an element of $L_2(Q; \sigma)$. The operator U maps $L_2(Q; \sigma)$ into itself because of the following norm-estimate which is analogous to (4.4):

$$\| wU\Phi \| \leqslant D \| w\Phi \| . \tag{4.8}$$

The constant $D = D(\sigma) > 0$ is determined by:

$$D: = \max | \mu - \mu_i | \cdot C, \qquad i \in T(m). \tag{4.9}$$

In analogy to C_0, the number $C = C(\sigma)$ is the maximum of

$$B: = |(- \sigma_1 + j\tau_1)^2 + (- \sigma_2 + j\tau_2)^2 + \mu |^{-1}$$

with respect to $\tau = (\tau_1, \tau_2) \in \mathbb{R} \times \mathbb{R}$. The operator U defines a contraction mapping on $L_2(Q; \sigma)$ if the maximum of D_0 and D becomes less than 1. This condition being fulfilled the solution of the equation (3.8) as element of L_2 is independent of μ due to Theorem 2. Therefore we consider the equation (3.8) only for the following subset of K_0:

$$K: = \{\mu \in K_0: \operatorname{Re} \mu + q_1{}^2 \leqslant 0\}. \tag{4.10}$$

The set K is not empty, and the contraction property of U on $L_2(Q; \sigma)$ results essentially from:

LEMMA 7. *For every $\mu \in K$ the number $D = D(\sigma)$ defines a continuous mapping in a neighbourhood of $\sigma = 0$.*

As the value of D at $\sigma = 0$ is equal to D_0, it follows from the above lemma that $D(\sigma)$ is less than 1 in a neighbourhood of $\sigma = 0$. For every $\mu \in K$ therefore a positive number $\delta = \delta(\mu) < q_1$ exists, which is such that U is a contractive mapping in all spaces $L_2(Q; \sigma)$ with $| \sigma | \leqslant \delta$ holding. This reasoning leads to:

THEOREM 4. *The unique solution $\Phi: Q \to \mathbb{C}$ of the equation* (3.8) *in L_2 has the following property: There exists a positive number δ which is such that Φ is square-integrable with the weight function $w: Q \to \mathbb{R}$ defined as $w(x) = \exp(\sigma x)$ for $x \in Q$, where $\sigma \in \mathbb{R} \times \mathbb{R}$ is restricted by $| \sigma | \leqslant \delta$.*

Using the preceding results the next section is devoted to the discussion of the transmission problem formulated above.

5. Existence and Uniqueness

We shall verify that the solution $\Phi: Q \to \mathbb{C}$ of the equation (3.8) satisfies all conditions of the problem quoted in Section 2. To begin with, it follows from Lemma 3 and Theorem 3 that Φ is continuous on $Q - \partial S_1$. The discontinuity of Φ on ∂S_1 is that of Ω^0. The differentiability properties of Φ are consequences of:

Lemma 8. *Let $G: Q \to \mathbb{C}$ be bounded. Then the convolution*

$$H: = \frac{1}{4j} H_0^{(1)} * G$$

exists, is bounded and uniformly continuous. The first-order partial derivatives exist, are continuous and result from differentiation under the integral sign:

$$D_l H = \frac{1}{4j} (D_l H_0^{(1)}) * G: Q \to \mathbb{C}, \qquad l = 1, 2.$$

If G is once continuously differentiable on an open subset Q_0 of Q, the second order partial derivatives exist and are continuous on Q_0. For every $x \in Q_0$ it holds:

$$(\Delta + k^2) H(x) = G(x).$$

We conclude from the above that Φ as a solution of the equation (3.8) is once continuously differentiable on $Q - \partial S_1$. The discontinuities of the first-order partial derivatives of Φ on ∂S_1 originate from Ω^0. The representation (4.3) of the operator U and the second part of Lemma 8 lead to the consequence that Φ is twice continuously differentiable on the open sector S_i, $i \in T(m)$. Here the corresponding differentiability properties of Ω^0, given by the primary field, must be involved. The fact that the restriction of Φ on S_i solves the Helmholtz equation, can now be verified by the following reasoning: It holds for Ω^0 on account of the primary field properties:

$$(\Delta + \mu_1)\, \Omega^0(x) = 0, \qquad \text{if} \qquad x \in Q - \partial S_1. \tag{5.1}$$

It results from Lemma 8:

$$(\Delta + \mu)\, \Omega^1(x) = (\mu_1 - \mu)\, \Omega^0(x), \qquad \text{if} \qquad x \in Q - \partial S_1. \tag{5.2}$$

The same is valid for the remaining additive terms on the right-hand side of (3.8):

$$(\Delta + \mu)(U\Phi)(x) = (\Gamma\Phi)(x), \qquad \text{if} \qquad x \in Q - \Sigma\partial S_i\,, \quad i \in T(m). \tag{5.3}$$

The three last equations lead to the conclusion:

$$(\Delta + \mu_i)\,\Phi(x) = 0, \qquad \text{if} \qquad x \in S_i\,, \quad i \in T(m). \tag{5.4}$$

The condition 2 of the problem quoted is fulfilled by the solution of the equation (3.8) on account of the properties of Ω^0 and the convolution. Since $\Phi\colon Q \to \mathbb{C}$ and its first order partial derivatives are bounded, the edge conditions of the problem are satisfied, if we choose α to be zero. In order to verify the radiation conditions we fix a positive number δ less than q_1 which is such that Theorem 4 is valid. Now we set $\delta = \beta$, the existence of which is required in the formulation of the transmission problem. If $\sigma \in \mathbb{R} \times \mathbb{R}$ and $|\sigma| \leqslant \beta$ hold, the following estimate is valid for every $x \in Q$ as a consequence of the equation (3.8) and Theorem 4:

$$\begin{aligned} |w\Phi| \leqslant \sup |w\Omega^0| + \tfrac{1}{4}\,\| wH_0^{(1)} \| \\ \cdot (|\mu_1 - \mu|\,\| w\Omega^0 \| + \max|\mu - \mu_i|\,\| w\Phi \|), \qquad i \in T(m). \end{aligned} \tag{5.5}$$

The right-hand side of the above inequality is bounded with respect to σ by a constant M which depends on β: $M = M(\beta)$. For any direction x_0 and $x = \rho x_0 \in Q$ the number $\sigma \in \mathbb{R} \times \mathbb{R}$, $|\sigma| \leqslant \beta$, can be chosen in such a way that σx is equal to $\beta\rho$. Hence the estimate (5.5) leads to:

$$e^{\beta\rho}\,|\Phi(\rho x_0)| \leqslant M(\beta).$$

This holds especially if ρ converges to infinity. Therefore the first radiation condition of the problem is satisfied by the solution of the equation (3.8). In order to verify the second condition of that kind, we take into account Lemma 8 and the fact that the first-order partial derivatives of $H_0^{(1)}$ are integrable on Q with the weight function w. Using (5.5) we arrive at:

$$e^{\beta\rho}\,|\operatorname{grad}\Phi(\rho x_0)| \leqslant N(\beta).$$

The constant $N(\beta)$ contains $M(\beta)$ and upper bounds for $\sup|w \operatorname{grad}\Omega^0|$ and for the Lebesgue integral of $|wD_lH_0^{(1)}|$, $l = 1, 2$. With respect to σ the bounds are to be taken on $|\sigma| \leqslant \beta$. The above estimate is valid especially if ρ goes to infinity. The preceding arguments lead to the conclusion that the problem quoted in Section 2 has at least one solution. Concerning the uniqueness the following reasoning holds: We assume the existence of at least two different solutions. It follows that for the difference of the two the integral equation (3.8) is valid Ω being zero. This equation has only the trivial solution. Summarizing the main result is as follows:

THEOREM 5. *The electromagnetic transmission problem formulated above has exactly one solution. It is characterized by the integral equations* (3.8) *being solvable by successive approximations.*

We conclude this section with some remarks on generalizations of the problem. The fixed-point method established seems to be applicable to transmission problems, which involve other partial differential equations than the Helmholtz one. This was suggested by Professor C. H. Wilcox to the author. Also the restriction on wedge-shaped regions is not decisive. The division of the plane into sectors may be replaced by a finite set of simply connected domains, the boundaries of which are the images of piecewise smooth curves of not necessarily finite length. This problem has been sketched in [21]. If, finally, the representation (4.3) of the operator U is considered, Lemma 8 leads to the conclusion that the solution of (3.8) solves the Helmholtz equation with a variable coefficient.

6. REMARKS ON THE LEMMAS QUOTED

1. The proof of Lemma 1 may be sketched as follows: The sector S can be exhausted by sets $S(r, R)$, $0 < r < R$, each of which being the intersection of S and the annulus between the circles of radius r and R centered at the origin. The boundary of $S(r, R)$ is the image of piecewise smooth curves. Using the continuity and differentiability conditions formulated one concludes that the formula (3.1) holds on $S(r, R)$. Due to the edge conditions the limits of both sides of the equation exist and are equal if r approaches zero. The same holds if R goes to infinity when studying the radiation conditions.

2. Lemma 2 is a consequence of Lemma 4 because the Laplace transform is holomorphic.

3. The justification of Lemma 3 proceeds in the following manner: The existence and boundedness of the convolution result from the Cauchy inequality. The convolution being uniformly continuous is a consequence of the continuity of the translation-operator in L_2. This property is proved in [10] in the case of square-integrable functions on $\mathbb{R}$. The other statements of Lemma 3 are proved by using the Fubini theorem.

4. The formula (3.6) of Lemma 4 is stated in [37]. The domain of validity follows from the property of $H_0^{(1)}$ if the independent variable approaches infinity. If $u \in \mathbb{C} \times \mathbb{C}$ is specialized on the pair of imaginary axes, the formula

(3.6) may be derived by transforming the Laplace integral to polar coordinates. Using an integral representation of the Bessel function J_0 the integral can be evaluated explicitly by special formulas given in [25]. Analytic continuation then leads to the domain of validity stated.

5. In order to prove Lemma 5 we use the convolution theorem and the Parseval identity of the Fourier–Plancherel transform. Then the formula (3.6) leads to the norm-estimate (4.1).

6. Lemma 6 results immediately from the generalized Parseval identity of the Fourier–Plancherel transform.

7. The continuity statement of Lemma 7 follows by considering the properties of

$$B: = |(-\sigma_1 + j\tau_1)^2 + (-\sigma_2 + j\tau_2)^2 + \mu|^{-1}.$$

On account of Lemma 2 B is different from zero for all $\tau = (\tau_1, \tau_2) \in \mathbb{R} \times \mathbb{R}$, if $\sigma = (\sigma_1, \sigma_2) \in \mathbb{R} \times \mathbb{R}$ is restricted by $0 \leqslant |\sigma| < q$, q being the imaginary part of the square root of μ. Therefore B defines a continuous and differentiable mapping on the τ-plane the values of which converge uniformly with respect to all directions to zero as $|\tau|$ goes to infinity. As a consequence of this fact B has an absolute maximum at $\bar{\tau} \in \mathbb{R} \times \mathbb{R}$. It holds $\bar{\tau} = \bar{\tau}(\sigma)$ and grad $B(\bar{\tau}) = 0$. By the last condition $\bar{\tau}$ is determined implicitly as a function of σ. It follows from the implicit function theorem that $\bar{\tau}$ defines a continuous mapping in the neighbourhood of $\sigma = 0$. The same is true for $C = \max B$. Observing (4.9) we arrive at the result quoted.

8. Proving Lemma 8, we observe at first that the Hankel function of the first kind of order zero is integrable as was stated in Lemma 4. From this property results that the defined convolution H exists and is bounded. H is uniformly continuous because of the continuity of the translation-operator in the space of integrable functions which fact is proved in [12]. The statements concerning the first-order partial derivatives are consequences of the fact that in the corresponding Newton quotients the limit sign can be put inside the integral.

In order to verify the second part of Lemma 8 we introduce the circle $K(x, \epsilon) \subset Q_0$ with radius $\epsilon > 0$ being centered at $x \in Q_0$. Then the convolution integrals defining the first-order partial derivatives may be decomposed additively into two by integrating separately over $K(x, \epsilon)$ and over $Q - K(x, \epsilon)$. The integrals over $Q - K(x, \epsilon)$ can be differentiated at x under the integral sign. The differentiation of the integrals taken over $K(x, \epsilon)$ is carried out by using the condition on G quoted. It follows that the second-order partial derivatives of H exist at $x \in Q_0$. The given partial differential equation results by transforming the integrals taken over $K(x, \epsilon)$ into integrals taken over $\partial K(x, \epsilon)$ by means of Green's theorem and then passing to the limit as ϵ converges to zero.

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